Reply to comment by F. A. Dahlen and G. Nolet on ‘On sensitivity kernels for ‘wave-equation’ transmission tomography’

Maarten V. de Hoop\textsuperscript{1,2} and Robert D. van der Hilst\textsuperscript{2}

\textsuperscript{1}Center for Computational and Applied Mathematics, Purdue University, West-Lafayette, IN 47907, USA. E-mail: mdehoop@purdue.edu
\textsuperscript{2}Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, Rm 54-522, Cambridge, MA 02139, USA

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\textbf{SUMMARY}
We thank Dahlen & Nolet for the comments (DN05) on our paper (HH05). There are many points of agreement, as we think is clear from HH05, but we respectfully continue to differ in opinion on some fundamental aspects of the finite frequency sensitivity kernels known as ‘banana doughnut’ kernels—hereinafter BDKs, as per the original nomenclature of Dahlen \textit{et al.}—and their benefit to global tomography. In contrast to DN05’s summary statement, HH05’s main concern about BDKs is not the effect of uncertainty in the earthquake source signature or origin time. HH05 argue that (i) the evaluation of sensitivity kernels in simple media has limitations for the interpretation of broad-band signals by means of (linearized) finite frequency tomography; (ii) finite frequency kernels are (indeed) oscillatory, but in general heterogeneity their structure will be complex and different from BD features; (iii) the resolved length scales of model variations are induced by the spectral scales present in the data, which makes the notion of ‘hole’ irrelevant; and (iv) with the need for ‘damping’ (regularization) and without a basis that matches properly the multi-scale aspects of finite frequency sensitivity, ray theory or finite frequency theory inversions are likely to yield results that are practically the same.

\textbf{Key words:} finite-frequency sensitivity kernels, transmission tomography.

1 BDKs: Theory Versus Application

Dahlen & Nolet 2005, (hereinafter DN05) reiterate that ‘the [BDKs] properly account for diffractive wavefront healing effects … and enable improved imaging of adequately sampled, small scale, mantle wavespeed anomalies’. Technical issues aside, whether or not this ‘improved imaging’ of ‘small scale … wavespeed anomalies’ has materialized yet is a matter of debate. In fact, Van der Hilst & De Hoop (2005) show that the models produced by Montelli \textit{et al.} (2004a,b) either with finite frequency theory (FFT) or ray theory (RT) are rather similar. This ‘Comment’ and ‘Reply’ dialogue on finite frequency theory, in general, and the technical merits of BDKs, in particular, should be viewed against this background.\textsuperscript{1}

Unless the heterogeneity is in the regime where ray theory is valid (and BDKs not formally needed), the similarity of RT and FFT models is perhaps somewhat surprising considering the large differences between the kernels implied in ray theory (imprint of infinitesimally narrow ray paths) and their finite frequency realizations (imprint of Fresnel volumes). Consistent with the analysis of the wave-equation transmission tomography kernels as distributions, the discretization of the problem using basis fuctions such as grids, voronoi cells, splines … may (in large parts of the mantle) not preserve the ‘doughnut’ hole characteristic for BDKs but, effectively, render ‘fat ray’ kernels with negative sensitivity on the ray (see HH05, their fig. 5a), not dissimilar from RT kernels. There is a tradeoff between the (finite frequency) regularization of the kernel and the damping of the inversion.

Furthermore, the similarity between RT and FFT images signals the need for theory or computational approaches that lead to more substantial image improvements. BDKs were developed to improve the interpretation of existing travel-time measurements. This was—and is—an important objective, but one should realize that such travel times represent a small fraction of the information contained in broad-band recordings of the seismic wavefield. We can agree or disagree on technicalities, but a methodology is required that precisely unravels the scales in model variation through a time-frequency, multi-resolution analysis of the waveform data.

2 Multi-Scale Aspects
We agree with Dahlen \textit{et al.} that finite-frequency waves sense structure (also) off the unperturbed source-receiver ray and that—\textit{in\textsuperscript{1}Note to readers familiar with (earlier) ‘on-line’ versions: this ‘Reply’ is different from our response to the original ‘Comment’ by Dahlen & Nolet. The original ‘Comment’ and ‘Reply’ addressed technical aspects of finite frequency theory as well as the effect of using BDKs on tomographic images, but at the request of Dahlen & Nolet (private communication, 2005) our views on the latter will now be published elsewhere (Van der Hilst & De Hoop 2005).
principle—scattering must be considered to image heterogeneity with length scales that are relatively small compared to the seismic wavelengths. This is represented in BDKs, but HH05 point out that the assumptions under which they can be considered reasonable renditions of the actual finite frequency sensitivity are quite restrictive.

In the linearized framework under consideration, the Green’s functions are calculated in a smooth, quasi-homogeneous background. This has important consequences for tomographic inversion: first, such kernels have no knowledge about the level of ‘diffactive wavefront healing’ in the data to which they are applied; secondly, since no explicit separation of scales has been applied to them, such kernels have no explicit knowledge of scales and types of heterogeneity (such as, plumes, slabs) through which they pass; third, the data only resolve structure at spatial scales that are represented in them. The sensitivity has (of course) an oscillatory nature and can, thus, be larger off than on the ray. But if, in linearized sense, the data do not contain the information needed to see a (smooth, finite scale) anomaly on the (source-receiver) ray, then such an anomaly will not be visible off the ray either. Conversely, the sensitivity does not vanish on the ray for scales represented in the data. In a space-scale sense the latter is consistent with the ‘fat man’ notion of Hung et al. (2001) that if you place a large enough anomaly on the ray the data would be sensitive to it.

The ultimate consequences of this scale concept are important, and they motivated HH05 to propose a different approach toward finite frequency imaging than the BD approach defended by DN05. In particular, to transform integral operators (defined by their kernels) into a matrix representation used in the tomographic inversions, we recognize a need for a basis—or frame—that properly matches the multi-scale aspects of finite frequency wave behavior—and that is consistent with the notion of wavefront propagation. HH05 introduce an explicit multi-resolution analysis based on curvelets (see Fig. 1), which provide a frame with the correct properties for finite-frequency wave equation tomography – and renders a natural analogue of the X-ray transform: (i) data resolve variations in structure primarily in directions perpendicular to the ray (except in the vicinity of the source and receiver); (ii) finite bandwidth data resolve only certain scales (that is, the number of annuli, or spatial scales, that contribute to the sensitivity is limited); (iii) sensitivity on the ray is not zero for scales represented in data, and that (iv) the higher the frequency, the smaller the length scale of the structures that can be resolved. Fig. 1 demonstrates that the resolution of structure at a certain scale is practically the same whether it is located on the unperturbed ray (i.e. in the ‘doughnut hole’) or away from it (e.g. on the BDK ring of maximum sensitivity). In other words, waves with a certain frequency content are (in)sensitive to certain scales whether they occur on the ray or away from it. A detailed analysis of the multi-resolution concept for wave equation tomography is beyond the scope of this ‘Reply’ and will be presented elsewhere.

### 3 MEASUREMENT

We agree with Dahlen et al. (2000) that the type of measurement should dictate which approach to (approximating) propagating waves should be used. In principle, one should not use ray theory to back project ‘finite-frequency travel times’ measured by time-domain.

As a minor issue, we note that this relationship is violated in some of Dahlen et al.’s comparisons of RT and FFT inversions through resolution tests with synthetic data. To discuss the relative merits of FFT over RT one should use data consistent with RT in the RT inversions and data consistent with BDKs main cross correlation. We just propose a different approach toward measuring and interpreting finite frequency data. But we did not state that the cross-correlation criterion approach to transmission tomography and the associated optimization scheme, as used by Dahlen et al. (2000), are incorrect. The cross-correlation/optimum criterion is, indeed, a ‘stable’ criterion to compare the ‘distance’ between two wavefields. But it is just a criterion, and it is only tied to the notion of travel time proper in the asymptotic broad-band limit of two delta waves (possibly convolved with a source or station signature). This is analyzed in detail in Hörmann & De Hoop (2002). One can give any name to the maximum of the cross correlation, such as ‘finite-frequency travel time shift’, but frequency dependent travel-time shifts have little meaning outside the context of an explicit multi-resolution, time-frequency analysis.

### 4 THEORETICAL CONSIDERATIONS

The basic theoretical observations of HH05 are that (i) the FFT kernels induced by the cross-correlation criterion are oscillatory (no surprise), the precise pattern of their zero crossings is determined by the background medium (but does not matter), and a zero crossing needs not coincide with the unperturbed ray paths, and that (ii) one should not view the kernels point-wise (Strichartz 2003) but in a multi-resolution framework (see Section 2).

#### 4.1 Ray kernel as an infinite bandwidth realization of finite frequency (wave) kernel

We assume that (i) the velocity model is smooth and that (ii) the sources can be represented by point sources and, hence, that the wavefield (that is, the displacement $u$ at receiver $r$) can be written as a time convolution of a source signature $W_{m}(t)$ with a Green’s function $G_{t}$ (that is, $u_{r} = W_{m} * G_{r}$). We do not claim that the further analysis applies to frequencies arbitrarily close to zero.

The key ingredient in the kernel development is the Green’s function in the unperturbed model. The Green’s function can be represented by oscillatory integrals (OIs)—see Duistermaat (1996) and Dencker (1982) for the scalar wave and elastic wave case, respectively. The leading-order amplitude in the OI yields a Maslov asymptotic representation of the Green’s function. The applicability of this is warranted under a ‘far-field’ approximation—but the full amplitude in the OI can be used as well. (NB. The analysis and arguments presented in HH05 hold for general OIs, but to keep the exposition transparent we used only the simplest form and left out the higher order contributions—HH05 eq. 1 then reduces to HH05 eq. 2.)

DN05 object to HH05’s remark that wave equation and travel time tomography share the same kernel. Of course we did not mean to imply that ‘banana doughnut type kernels’ are the same as ‘ray paths’. But starting from the Green’s functions representation of the kernel, we showed that the kernel tends to the travel time tomography kernel as the frequency (bandwidth) becomes infinite. Conversely, the band-limited Maslov representations of the Green’s functions in the kernel capture its leading finite-frequency features. In this sense, ‘travel time’ and ‘wave-equation’ tomography do indeed share the same kernel. This is consistent with the observation that asymptotically (that is, ‘delta waves’) the maximum of the cross correlation yields the travel time shift. Indeed, in the multi-resolution setting in the case of FFT inversions. M04a test the RT inversions on synthetic delay times that are consistent with the FFT cross correlation. These data do not contain the scales assumed in RT.

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Figure 1. This figure illustrates the scale and directional content of a kernel (see HH05 for a more detailed discussion). Left: space domain representation; right: Fourier domain representation. The space domain depicts the contributions in space to the (normalized) sensitivity kernel, which is projected onto a curvelet frame. In the Fourier domain representation the largest length scales (i.e. smallest wave numbers $k$) plot toward the center, and the direction in which the scale annuli are filled mark the direction of maximum resolution. In this example, the kernel is of ‘banana doughnut’ type and the (background) medium is, thus, homogeneous. The top row (same as Fig. 8d of HH05) illustrates contributions (right) to the kernel from curvelets located on the unperturbed ray half-way between source and receiver (indicated with a cross in the space domain panel). The Fourier transform shows that data with the specific frequency content used here do sense structure on the unperturbed ray but only for certain scales, and (primarily) in directions orthogonal to the ray. In the bottom row the spatial location of curvelets considered is moved away from the ray to the first ‘optimum’ of the regularized kernel (lower left). The Fourier domain representation, and hence the ‘curvelet content’, is practically the same as for the point on the ray. This demonstrates that data with a certain frequency content are sensitive to structure at certain length scales (wave numbers) and that this sensitivity is the same for structure on the ray and away from it.

mentioned in Section 2 the behavior of the kernel can be described correctly all the way from the narrow-band situation to the infinite bandwidth limit.

DN05 seem confused about the notions of distribution and its regularization. Separating out source and receiver signatures, the exact kernel would be a distribution built from Green’s functions (solutions of wave equations), which are (Lagrangian) distributions themselves. The instrument response acts as an operator on the Green’s function in the receiver coordinates while the source appears as an initial value in the source coordinates. The approximate, finite bandwidth kernel is a (smooth) regularization of this distribution but should not be considered a distribution itself. Of course, the kernel, as well as its adjoint, define the integral operators relevant to the imaging.

Finally, the singular support of the kernel coincides with the unperturbed ray. To analyze what happens on this ray (for example, upon regularization with a source signature) it suffices to focus on the singular part of the kernel, which can be derived from the Maslov representations of the Green’s functions. In fact, these representations allow the development of fast finite frequency methods, even for complex media.

4.2 Caustics and kernel complexity

HH05 agree with DN05 that under certain assumptions the (regularizations of the) kernels reveal a ‘banana doughnut’ behavior. Descriptions of and statements relating to the behavior of a kernel of a (locally linearized) inversion procedure should be generic, however, and not be tied to a particular, simple (background) medium. DN05 and HH05 concur that caustics affect the locations of the zero crossings in the point-wise evaluation of the kernels. What should be realized, however, is that caustics form readily and ubiquitously
due to smooth heterogeneities in the medium (see White; cf HH05) and can thus be expected to occur throughout Earth’s mantle (both between the source and scatter point and between the receiver and the scatterer). Caustics and scattering can effect the pulse shape of the phases that are subjected to the waveform cross-correlation and should thus not be ignored in the theories and kernel evaluations under consideration. Accounting only for the PP caustic, for example, is, therefore, not sufficient. In this context we argue that the use of BDKs will merely give a false sense of improvement when imaging complex media, such as the upper mantle, volcano interiors, the shallow subsurface, etc., because the actual ‘kernels’ may not even remotely resemble BDK features.

4.3 Errors in source signature

DN05 devote a significant fraction of their ‘Comment’ on this aspect. First, it is a minor issue (and not our main objection against the BD features). Secondly, HH05 do not suggest that one can—or should—account for errors in source signature by modifying the Frechet kernels; we merely pointed out that such (unknown) errors can render this effect altogether. If the parameter (error in) source signature is known a deconvolution can remove non-zero values of the actual sensitivity. Of course, the (unknown) errors can render the scales represented in them (implied by frequency range) to the spatial scales in the physics (or the ‘model’). We argue that the use of individual measurements of ballistic waves (i.e. phase arrivals) combined with back projection along a (linearized) kernel will not meet this challenge and that, instead, a rigorous broad-band, space-wavenumber multi-scale analysis is needed. HH05 and de Hoop & Van der Hilst (under revision for GJI, 2005) introduce approaches for wave equation transmission and reflection tomography based on multi resolution and adjoint state methods, whereas also computational full-wave approaches (e.g. Tromp et al. 2005) hold significant promise.

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REFERENCES


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